## Deriving the Binomial Theorem using Maclaurin expansions

We want to work out the expansion for $(a+b)^{n}$, so we would have to find it's derivative. In this case, $b$ could be a function of $a$, and this is very difficult to deal with. If we instead factored out an $a^{n}$, to get the below expression, we could perhaps simplify things:

$$
(a+b)^{n}=\left(a\left(1+\frac{b}{a}\right)\right)^{n}=a^{n}\left(1+\frac{b}{a}\right)^{n}
$$

$a^{n}$ is already expanded, so we're only interested in the expansion of the parenthesis. If we could come up with a formula for expanding parentheses of the form $(1+x)^{n}$, we could just plug in $\frac{b}{a}$ to work out an expansion for our original expression. So, we want to construct a Maclaurin series for $(1+x)^{n}$. I might also note that the binomial theorem only concerns integer powers, so we can assume n is an integer.

## Finding the $n$th derivative of $(1+x)^{n}$

We need to find a pattern in the derivatives. Let's take a few derivatives to see if we can notice anything:

$$
\begin{gathered}
f(x)=(1+x)^{n} \\
f^{\prime}(x)=n(1+x)^{n-1} \\
f^{\prime \prime}(x)=n(n-1)(1+x)^{n-2} \\
\vdots \\
f^{n-2}(x)=n(n-1)(n-2) \ldots 4 \cdot 3(1+x)^{2} \\
f^{n-1}(x)=n(n-1)(n-2) \ldots 3 \cdot 2(1+x) \\
f^{n}(x)=n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1 \\
f^{n+1}(x)=0
\end{gathered}
$$

Any derivatives beyond $n+1$ will be zero, since the nth derivative is just a constant.
If we look closely, we can see that these chains of $n(n-1)(n-2) \ldots$ can be described using factorials. We see that for every derivative we have the factorial corresponding to the order of that derivative, but we're missing n minus the order of the derivative number of factors. This means we can express the nth order derivative of $f(x)=(1+x)^{k}$ using division as:

$$
f^{n}(x)=\frac{k!}{(k-n)!}(1+x)^{k-n}
$$

Notice that the above only holds if n and k are integers and

$$
0 \leq n \leq k
$$

## Constructing the Maclaurin series

We will begin by looking at the Maclaurin expansion:

$$
f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^{n}=f(0)+f^{\prime}(0) x+\frac{f^{\prime \prime}(2)}{2!} \ldots
$$

Note that in the case of $(1+x)^{k}$ we only care about terms up to k , since all the derivatives after that will be zero. This gives the following expansion:

$$
(1+x)^{k}=\sum_{n=0}^{k} \frac{f^{n}(0)}{n!} x^{n}
$$

Now all we need is to work out the value for 0 of all the derivatives. We just derived a formula for the nth derivative, and it holds for just this interval:

$$
f^{n}(x)=\frac{k!}{(k-n)!}(1+x)^{k-n}
$$

Plugging in 0 , we get:

$$
f^{n}(0)=\frac{k!}{(k-n)!}(1+0)^{k-n}=\frac{k!}{(k-n)!} 1^{k-n}=\frac{k!}{(k-n)!}
$$

We can plug in this back into the expansion above:

$$
(1+x)^{k}=\sum_{n=0}^{k} \frac{\frac{k!}{(k-n)!}}{n!} x^{n}=\sum_{n=0}^{k} \frac{k!}{(k-n)!n!} x^{n}
$$

Now that we have the expansion for $(1+x)^{k}$, we can go back to the original problem of expanding $(a+b)^{n}$. We know from earlier:

$$
(a+b)^{n}=a^{n}\left(1+\frac{b}{a}\right)^{n}
$$

We can expand the parenthesis using our formula:

$$
\begin{gathered}
a^{n}\left(1+\frac{b}{a}\right)^{n}=a^{n}\left(\sum_{p=0}^{n} \frac{n!}{(n-p)!p!}\left(\frac{b}{a}\right)^{p}\right)=\sum_{p=0}^{n} \frac{n!}{(n-p)!p!} \cdot \frac{b^{p}}{a^{p}} \cdot a^{n}= \\
(a+b)^{n}=\sum_{p=0}^{n} \frac{n!}{(n-p)!p!} a^{n-p} b^{p}=\sum_{p=0}^{n}\binom{n}{p} a^{n-p} b^{p}
\end{gathered}
$$

Which is the Binomial Theorem.

## Proving the power rule without using the Binomial Theorem

The power rule (which was used to compute the above derivatives) is often proven using the Binomial Theorem. In the sake of avoiding a circular argument (proving the Binomial Theorem using the Binomial Theorem isn't very solid...), I will show an alternate (and in my opinion more elegant) proof of the power rule:

$$
\text { Let } y=x^{n}
$$

Taking the natural log of both sides gives:

$$
\ln (y)=\ln \left(x^{n}\right)
$$

Using the log identity $\log _{b}\left(x^{y}\right)=y \log _{b}(x)$ :

$$
\begin{gathered}
\ln (y)=n \ln (x) \\
\frac{d}{d x}[\ln (y)]=\frac{d}{d x}[n \ln (x)] \\
\frac{1}{y} \cdot \frac{d y}{d x}=\frac{n}{x} \\
\frac{d y}{d x}=\frac{n}{x} \cdot y
\end{gathered}
$$

Substituting in $y=x^{n}$ :

$$
\begin{gathered}
\frac{d y}{d x}=\frac{n}{x} \cdot x^{n} \\
\frac{d y}{d x}=n x^{n-1} \\
\text { Since } y=x^{n}: \\
\frac{d}{d x}\left[x^{n}\right]=n x^{n-1}
\end{gathered}
$$

Which is what we wanted to prove.

